On some Strong Ratio Limit Theorems for Heat Kernels

Dedicated to Louis Nirenberg on the occasion of his 85th birthday

MARTIN FRAAS

Department of Physics,
Technion - Israel Institute of Technology,
Haifa, Israel

DAVID KREJČIŘÍK

Department of Theoretical Physics,
Nuclear Physics Institute,
Academy of Sciences, 25068 Řež, Czech Republic

YEHUDA PINCHOVER

Department of Mathematics,
Technion - Israel Institute of Technology,
Haifa, Israel

ABSTRACT. We study strong ratio limit properties of the quotients of the heat kernels of subcritical and critical operators which are defined on a noncompact Riemannian manifold.

1. Introduction

Let M be a connected noncompact Riemannian manifold, and let $k_P^M(x,y,t)$ be the positive minimal (Dirichlet) heat kernel associated with the parabolic equation

$$u_t + Pu = 0 \qquad \text{on} \qquad M \times (0,\infty)\,, \tag{1}$$

where P is a second-order elliptic differential operator on M. The coefficients of P are assumed to be real but P is not necessarily symmetric. By definition, $(x,t) \mapsto k_P^M(x,y,t)$ is the minimal positive solution of (1), subject to the initial data δ_y, the Dirac distribution at $y \in M$. We say that the operator P is *subcritical* (respectively, *critical*) in M if for some $x \neq y$

$$\int_0^\infty k_P^M(x,y,\tau)\,\mathrm{d}\tau < \infty \qquad \left(\text{respectively, } \int_0^\infty k_P^M(x,y,\tau)\,\mathrm{d}\tau = \infty\right). \tag{2}$$

1991 *Mathematics Subject Classification.* Primary: 35K08; Secondary: 35B09, 58J35, 60J60.
Key words and phrases. Heat kernel, large time behavior, positive solutions, ratio limit theorem.

This paper coincides with that published in Discrete Contin. Dynam. Systems A **28** (2010), 495–509, except for Remark 4 added after the paper has appeared.

In this paper we are concerned with the large time behavior of the heat kernel k_P^M with regards to the criticality versus subcriticality property of the operator P. Since for any fixed $x,y\in M$, $x\neq y$, we have that $k_P^M(x,y,\cdot)\in L^1(\mathbb{R}_+)$ if and only if P is subcritical, it is natural to conjecture that *under some assumptions* the heat kernel of a subcritical operator P_+ in M decays (in time) faster than the heat kernel of a critical operator P_0 in M. More precisely, we are interested to study the following conjecture.

Conjecture 1. *Let P_+ and P_0 be respectively subcritical and critical operators in M. Then*

$$\lim_{t\to\infty}\frac{k_{P_+}^M(x,y,t)}{k_{P_0}^M(x,y,t)}=0 \tag{3}$$

locally uniformly in $M\times M$.

The relevance of this conjecture becomes clearer if we recall the relationship of (2) to properties of positive solutions of the elliptic equation

$$Pu=0 \qquad \text{on} \qquad M. \tag{4}$$

Denote the cone of all positive (weak) solutions of (4) by $\mathcal{C}_P(M)$. The *generalized principal eigenvalue* of P in M is defined by

$$\lambda_0=\lambda_0(P,M):=\sup\{\lambda\in\mathbb{R}\mid \mathcal{C}_{P-\lambda}(M)\neq\emptyset\}\,. \tag{5}$$

Throughout this paper we always assume that

$$\lambda_0\geq 0$$

(actually, as it will become clear below, it is enough to assume that $\lambda_0>-\infty$).

Recall that if $\lambda<\lambda_0$, then $P-\lambda$ is subcritical in M, and for $\lambda\leq\lambda_0$, we have $k_{P-\lambda}^M(x,y,t)=\mathrm{e}^{\lambda t}k_P^M(x,y,t)$. It follows that $\lambda_0(P_0,M)=0$ for any critical operator P_0 in M, while $\lambda_0(P_+,M)\geq 0$ for any subcritical operator P_+ in M.

It is well known that if P is subcritical in M, then P admits a *positive minimal Green function* $G_P^M(x,y)$ which is given by

$$G_P^M(x,y)=\int_0^\infty k_P^M(x,y,\tau)\,\mathrm{d}\tau. \tag{6}$$

On the other hand, if P is critical in M, then P does not admit a positive minimal Green function, but admits a distinguished *unique* positive solution $\varphi\in\mathcal{C}_P(M)$ satisfying $\varphi(x_0)=1$, where $x_0\in M$ is a reference point. Such a solution is called a *ground state* of the operator P in M [**1, 16, 21**]. A ground state is characterized by being a global positive solution of the equation $Pu=0$ on M of *minimal growth in a neighborhood of infinity in* M [**1**]. On the other hand, if P is subcritical in M, then for any fixed $y\in M$, the positive minimal Green function $G_P^M(\cdot,y)$ is a positive solution of the equation $Pu=0$ on $M\setminus\{y\}$ of minimal growth in a neighborhood of infinity in M. We also note that P is critical in M if and only the equation $Pu=0$ on M admits (up to a multiplicative constant) a *unique positive supersolution*. Furthermore, P is critical (respectively, subcritical) in M, if and only if P^* (the formal adjoint of P) is critical (respectively, subcritical) in M. The ground state of P^* is denoted by φ^*.

A critical operator P is said to be *positive-critical* in M if $\varphi^*\varphi\in L^1(M)$, and *null-critical* in M if $\varphi^*\varphi\notin L^1(M)$. The large time behavior of the heat kernel of a *general* elliptic operator P (with $\lambda_0\geq 0$) is governed by the following theorem.

Theorem 1.1 ([16, 18])**.** *Let $x, y \in M$. Then*

$$\lim_{t\to\infty} \mathrm{e}^{\lambda_0 t} k_P^M(x,y,t) = \begin{cases} \dfrac{\varphi(x)\varphi^*(y)}{\int_M \varphi(z)\varphi^*(z)\,\mathrm{d}\mu(z)} & \text{if } P-\lambda_0 \text{ is positive-critical},\\ 0 & \text{otherwise}. \end{cases}$$

Furthermore,

$$\lim_{t\to\infty} \mathrm{e}^{\lambda_0 t} k_P^M(x,y,t) = \lim_{\lambda\nearrow\lambda_0} (\lambda_0-\lambda) G_{P-\lambda}^M(x,y). \tag{7}$$

As a consequence of this theorem, we see that $\lim_{t\to\infty} \mathrm{e}^{\lambda_0 t} k_P^M(x,y,t)$ always exists. On the other hand, heat kernels might have slow decay (see for example [4] and the references therein). Therefore, it is natural to ask how fast versus slow this limit is approached, and in particular, to examine the validity of Conjecture 1. We note that Theorem 1.1 implies that Conjecture 1 obviously holds true if P_0 is positive-critical.

In [12, Theorems 4.2 and 4.4] M. Murata obtained the exact asymptotic for the heat kernels of nonnegative Schrödinger operators with *short-range* (real) potentials defined on $\mathbb{R}^d$, $d \geq 1$. These results imply that Conjecture 1 holds true for such operators.

The aim of the present paper is to discuss Conjecture 1 and closely related problems in the *general* case, and to obtain some results under minimal assumptions.

Our study is motivated by a recent paper [9] by D. Krejčiřík and E. Zuazua, where it is conjectured that for selfadjoint subcritical and critical operators P_+ and P_0, respectively, defined on $L^2(M, \mathrm{d}x)$ one has

$$\lim_{t\to\infty} \frac{\|\mathrm{e}^{-P_+ t}\|_{L^2(M,W\,\mathrm{d}x)\to L^2(M,\mathrm{d}x)}}{\|\mathrm{e}^{-P_0 t}\|_{L^2(M,W\,\mathrm{d}x)\to L^2(M,\mathrm{d}x)}} = 0 \tag{8}$$

for some positive weight function W. In fact, the above conjecture is proved in [9] for the Dirichlet Laplacian defined on a special class of quasi-cylindrical domains.

It turns out that Conjecture 1 is related to the following conjecture raised by E. B. Davies [6] in the self-adjoint case.

Conjecture 2 (Davies' Conjecture)**.** *Let $Lu = u_t + P(x,\partial_x)u$ be a parabolic operator which is defined on a noncompact Riemannian manifold M. Fix reference points $x_0, y_0 \in M$. Then*

$$\lim_{t\to\infty} \frac{k_P^M(x,y,t)}{k_P^M(x_0,y_0,t)} = a(x,y) \tag{9}$$

exists and is positive for all $x, y \in M$, Moreover, for any fixed $y \in M$ we have $a(\cdot, y) \in \mathcal{C}_{P-\lambda_0}(M)$. Similarly, for a fixed $x \in M$ we have $a(x,\cdot) \in \mathcal{C}_{P^*-\lambda_0}(M)$ (see also [19] and the references therein).

Remark 1. Obviously, Davies' Conjecture holds if P is positive-critical. Moreover, it holds true in the symmetric case (for a precise definition of P being symmetric see Section 2) if $\dim \mathcal{C}_P(M) = 1$ [3, Corollary 2.7]. In particular, it holds true for a critical symmetric operator. For a probabilistic interpretation of Conjecture 2, see [3].

On the other hand, G. Kozma announced [8] that he constructed a graph G such that for some two vertices $x, y \in G$ the sequence $\{k(x,x,n)/k(y,y,n)\}_{n=1}^{\infty}$ of the ratio of the corresponding heat kernel does not converge as $n \to \infty$.

The organization of this paper is as follows. In the following section, we give a precise definition of the operator P in M and introduce the necessary background to study Conjecture 1. In Section 3, we prove (under some additional assumptions) Conjecture 1 in the symmetric case (Theorem 3.1). In particular, Theorem 3.1 provides an affirmative answer to the conjecture in the case of positive perturbations (Corollary 1). The relationship between Davies' conjecture and Conjecture 1 is examined for nonsymmetric operators in Section 4. Two regimes are considered: positive perturbations (Theorem 4.1) and semismall perturbations (Theorem 4.2). We conclude the paper in Section 5, where we ask a general question concerning the equivalence of heat kernels on Riemannian manifolds and provide sufficient conditions for the validity of a principal hypothesis of theorems 2.3, 3.1, and 4.2.

2. Preliminaries

Let M be a smooth connected noncompact Riemannian manifold of dimension d. We recall the definition of a weighted manifold associated with M. Denote by $\mathrm{d}x$ the Riemannian density on M. The divergence and gradient with respect to the Riemannian metric on M are denoted by div and ∇, respectively. Let m be a positive measurable function on M such that m and m^{-1} are bounded on any compact subset of M. Set $\mathrm{d}\mu := m\mathrm{d}x$. The couple $(M, \mathrm{d}\mu)$ is called a *weighted manifold* over which we consider the Lebesgue spaces $L^p(M, \mathrm{d}\mu)$.

We associate to M *an exhaustion*, i.e. a sequence of smooth, relatively compact domains $\{M_j\}_{j=1}^{\infty}$ such that $M_1 \neq \emptyset$, $\overline{M}_j \subset M_{j+1}$ and $\cup_{j=1}^{\infty} M_j = \Omega$. For every $j \geq 1$, we denote $M_j^* := \Omega \setminus \overline{M_j}$.

We consider a second-order elliptic differential operator P which is defined on $(M, d\mu)$ by

$$Pu := -m^{-1}\mathrm{div}(mA\nabla u - muC) - \langle B, \nabla u\rangle + Du, \tag{10}$$

where D is a real-valued measurable function on M, B and C are measurable vector fields on M, and A is a symmetric locally bounded measurable section on M of $\mathrm{End}(TM)$ such that P is locally uniformly elliptic on M. Here T_xM, TM, $\mathrm{End}(T_xM)$ and $\mathrm{End}(TM)$ denote the tangent space to M at $x \in M$, the tangent bundle, the endomorphisms on T_xM and the corresponding bundle, respectively. The inner product and the induced norm on TM is denoted by $\langle\cdot,\cdot\rangle$ and $|\cdot|$, respectively. We assume that $D, |B|^2, |C|^2 \in L^p_{\mathrm{loc}}(M, \mathrm{d}\mu)$ for some $p > \max\{n/2, 1\}$.

We say that P is *symmetric* if $B = C = 0$ on M. So, in the symmetric case P has the form

$$Pu = -m^{-1}\mathrm{div}(mA\nabla u) + Du. \tag{11}$$

The reason for this terminology is that the minimal operator constructed from the formal differential operator (11), *i.e.* the restriction of P to $C_0^{\infty}(M)$, is symmetric in $L^2(M, \mathrm{d}\mu)$. The Friedrichs extension of the minimal operator defines a self-adjoint operator in $L^2(M, \mathrm{d}\mu)$; it acts weakly as (11) and satisfies Dirichlet boundary conditions on ∂M in a generalized sense. By definition, it is the operator associated with the closure of the quadratic form Q in $L^2(M, \mathrm{d}\mu)$ defined by

$$Q[u] := \int_M \left(\langle A\nabla u, \nabla u\rangle + D|u|^2\right) \mathrm{d}\mu \qquad u \in C_0^{\infty}(M)\,. \tag{12}$$

It is well known that for such operators we have

$$\lambda_0 = \inf \left\{ Q[u] \,\middle|\, u \in C_0^\infty(M),\ \int_M |u|^2 \,\mathrm{d}\mu = 1 \right\},$$

where λ_0 is the generalized principal eigenvalue of P introduced in (5). In other words, λ_0 equals to the bottom of the spectrum of the Friedrichs extension if P is symmetric.

Remark 2. Let $t_n \to \infty$. By a standard parabolic argument, we may extract a subsequence $\{t_{n_k}\}$ such that for every $x, y \in M$ and $s < 0$

$$a(x,y,s) := \lim_{k\to\infty} \frac{k_P^M(x,y,s+t_{n_k})}{k_P^M(x_0,y_0,t_{n_k})} \tag{13}$$

exists. Moreover, $a(\cdot,y,\cdot) \in \mathcal{H}_P(M\times\mathbb{R}_-)$, where $\mathcal{H}_P(M\times(a,b))$ denotes the cone of all nonnegative solutions of the equation (1) in $M\times(a,b)$. Note that in the selfadjoint case, the above is valid for all $s\in\mathbb{R}$ [**19**].

Now we recall some auxiliary results which we will need in the sequel. First, we mention convexity properties of heat kernels.

Lemma 2.1. *Consider the following one-parameter family of elliptic operators*

$$P_\alpha := P + \alpha V \qquad 0 \le \alpha \le 1,$$

where V is a nonzero potential. Assume that $\lambda_0(P_\alpha, M) \ge 0$ for $\alpha = 0,1$. Then $\lambda_0(P_\alpha, M) \ge 0$ for $0 \le \alpha \le 1$, and the corresponding heat kernels satisfy the inequality

$$k_{P_\alpha}^M(x,y,t) \le [k_{P_0}^M(x,y,t)]^{1-\alpha}[k_{P_1}^M(x,y,t)]^{\alpha} \quad \forall\, x,y\in M,\ t>0,\ 0\le\alpha\le 1. \tag{14}$$

Moreover, P_α is subcritical for any $0 < \alpha < 1$.

For a proof of the lemma see [**15**]. In particular, (14) is proved by applying Hölder's inequality to the Feynmann-Kac formula (see e.g., [**22**, Lemma B.7.7]).

We also need the following key lemma

Lemma 2.2. *Assume that $\lambda_0(P, M) \ge 0$, and that either P is symmetric or that Davies' conjecture holds for P in M. Then for any fixed $x, y \in M$ we have*

$$\lim_{t\to\infty} \frac{k_P^M(x,y,\tau+t)}{k_P^M(x,y,t)} = \mathrm{e}^{-\lambda_0\tau} \qquad \forall \tau \in \mathbb{R}_-. \tag{15}$$

Proof. If P is symmetric, then the function $t \mapsto k_P^M(x,x,t)$ is log-convex, and therefore the lemma follows by a polarization argument (see for example [**5, 6**]).

Suppose now that Davies' conjecture holds for P in M. Then as in the proof of [**19**, Theorem 3.1], fix $y\in M$ and let $\{t_n\}$ be a sequence such that $t_n\to\infty$. Consider the sequence $\{\frac{k_P^M(x,y,\tau+t_n)}{k_P^M(y,y,t_n)}\}$ that converges (up to a subsequence) to a nonnegative solution $K_P^M(x,\tau) \in \mathcal{H}_P(M\times\mathbb{R}_-)$ (see Remark 2). By our assumption, for any τ we have

$$\lim_{n\to\infty} \frac{k_P^M(x,y,\tau+t_n)}{k_P^M(y,y,\tau+t_n)} = \lim_{s\to\infty} \frac{k_P^M(x,y,s)}{k_P^M(y,y,s)} = \frac{a(x,y)}{a(y,y)} =: b(x) > 0,$$

where $b \in \mathcal{C}_{P-\lambda_0}(M)$, and b does not depend on the sequence $\{t_n\}$.

On the other hand,

$$\lim_{n\to\infty} \frac{k_P^M(y,y,\tau+t_n)}{k_P^M(y,y,t_n)} = K_P^M(y,\tau) =: f(\tau).$$

Since

$$\frac{k_P^M(x,y,\tau+t_n)}{k_P^M(y,y,t_n)} = \frac{k_P^M(x,y,\tau+t_n)}{k_P^M(y,y,\tau+t_n)} \cdot \frac{k_P^M(y,y,\tau+t_n)}{k_P^M(y,y,t_n)},$$

we have

$$K_P^M(x,\tau) = b(x)f(\tau).$$

Since $K_P^M(x,\tau)$ solves the parabolic equation $u_\tau + Pu = 0$ in $M \times \mathbb{R}_-$, and $b \in \mathcal{C}_{P-\lambda_0}(M)$, it follows that f solves the initial value problem (backwards in time)

$$f' + \lambda_0 f = 0 \quad \text{ on } \quad \mathbb{R}_-, \quad f(0) = 1.$$

In particular, f does not depend on the sequence $\{t_n\}$. Thus,

$$\lim_{t\to\infty} \frac{k_P^M(y,y,\tau+t)}{k_P^M(y,y,t)} = f(\tau) = \mathrm{e}^{-\lambda_0\tau}.$$

Finally,

$$\lim_{t\to\infty} \frac{k_P^M(x,y,\tau+t)}{k_P^M(x,y,t)} = \lim_{t\to\infty} \frac{k_P^M(y,y,\tau+t)}{k_P^M(y,y,t)} \cdot \frac{k_P^M(x,y,\tau+t)}{k_P^M(y,y,\tau+t)} \cdot \frac{k_P^M(y,y,t)}{k_P^M(x,y,t)}$$

$$= \mathrm{e}^{-\lambda_0\tau} \cdot b(x) \cdot (b(x))^{-1} = \mathrm{e}^{-\lambda_0\tau}.$$

□

It turns out that Lemma 2.2 implies that the case $\lambda_0(P_+, M) > 0$ is easier than the case $\lambda_0(P_+, M) = 0$. Moreover, if $\lambda_0(P_+, M) > 0$, then the assumptions that we need for the validity of Conjecture 1 are weaker. We have

Theorem 2.3. *Let P_0 be critical operator in M, and let P_+ be a subcritical operator in M satisfying $\lambda_+ := \lambda_0(P_+, M) > 0$. Suppose that either P_0 and P_+ are symmetric operators, or that Davies' conjecture (Conjecture 2) holds true for both $k_{P_0}^M$ and $k_{P_+}^M$.*

Assume further that for some fixed $y_1 \in M$ there exists a positive constant C satisfying the following condition: for each $x \in M$ there exists $T(x) > 0$ such that

$$k_{P_+}^M(x,y_1,t) \leq C k_{P_0}^M(x,y_1,t) \qquad \forall t > T(x). \tag{16}$$

Then

$$\lim_{t\to\infty} \frac{k_{P_+}^M(x,y,t)}{k_{P_0}^M(x,y,t)} = 0 \tag{17}$$

locally uniformly in $M \times M$.

Proof. Fix $x \in M$, and $s \in \mathbb{R}_-$, and let $y_1 \in M$ be the point satisfying (16). We have

$$\frac{k_{P_+}^M(x,y_1,t)}{k_{P_0}^M(x,y_1,t)} = \frac{k_{P_+}^M(x,y_1,t+s)}{k_{P_0}^M(x,y_1,t+s)} \times \frac{k_{P_+}^M(x,y_1,t)}{k_{P_+}^M(x,y_1,t+s)} \times \frac{k_{P_0}^M(x,y_1,t+s)}{k_{P_0}^M(x,y_1,t)}. \tag{18}$$

Recall that $\lambda_0(P_0, M) = 0$, and by our assumption $\lambda_+ > 0$. By Lemma 2.2 we have

$$\lim_{t\to\infty} \frac{k_{P_+}^M(x,y_1,t)}{k_{P_+}^M(x,y_1,t+s)} = e^{\lambda_+ s}, \qquad \lim_{t\to\infty} \frac{k_{P_0}^M(x,y_1,t+s)}{k_{P_0}^M(x,y_1,t)} = 1. \tag{19}$$

Therefore, using (19) and our assumption (16), it follows from (18) that for t sufficiently large we have

$$\frac{k_{P_+}^M(x,y_1,t)}{k_{P_0}^M(x,y_1,t)} \leq 2Ce^{\lambda_+ s}. \tag{20}$$

Since s is an arbitrary negative number, (20) implies that

$$\lim_{t\to\infty} \frac{k_{P_+}^M(x,y_1,t)}{k_{P_0}^M(x,y_1,t)} = 0. \tag{21}$$

The parabolic Harnack inequality and a standard parabolic regularity argument imply now that

$$\lim_{t\to\infty} \frac{k_{P_+}^M(x,y,t)}{k_{P_0}^M(x,y,t)} = 0$$

locally uniformly in $M \times M$. □

By the generalized maximum principle, assumption (16) is satisfied with $C = 1$ if $P_+ = P_0 + V$ and V is any *nonnegative* potential. In Section 5, we discuss some other conditions under which assumption (16) is satisfied.

We shall need also the following Liouville comparison theorem (see [**20**]).

Theorem 2.4. *Let P_0 and P_1 be two symmetric operators defined on M of the form (11). Assume that the following assumptions hold true.*

(i) *The operator P_0 is critical in M. Denote by $\varphi \in \mathcal{C}_{P_0}(M)$ its ground state.*
(ii) *$\lambda_0(P_1, M) \geq 0$, and there exists a real function $\psi \in H^1_{\mathrm{loc}}(M)$ such that $\psi_+ \neq 0$, and $P_1\psi \leq 0$ in M, where $u_+(x) := \max\{0, u(x)\}$.*
(iii) *Denote by A_1, A_0 the sections on M of* $\mathrm{End}(TM)$, *and by m_1, m_0 the weights corresponding to P_1, P_0, respectively. The following matrix inequality holds*

$$(\psi_+)^2(x)m_1(x)A_1(x) \leq C\varphi^2(x)m_0(x)A_0(x) \qquad \text{for a.e. } x \in M, \tag{22}$$

where $C > 0$ is a positive constant.

Then the operator P_1 is critical in M, and ψ is its ground state. In particular, $\dim \mathcal{C}_{P_1}(M) = 1$ *and* $\lambda_0(P_1, M) = 0$.

Let $f, g \in C(\Omega)$ be nonnegative functions, we use the notation $f \asymp g$ on Ω if there exists a positive constant C such that

$$C^{-1}g(x) \leq f(x) \leq Cg(x) \qquad \text{for all } x \in \Omega.$$

In the sequel we shall need also to use results concerning small and semismall perturbations. These notions were introduced in [**14**] and [**13**] respectively, and are closely related to the stability of $\mathcal{C}_P(\Omega)$ under perturbation by a potential V.

Definition 2.5. Let P be a subcritical operator in M, and let V be a potential defined on M.

(i) We say that V is a *small perturbation* of P in M if

$$\lim_{j\to\infty} \left\{ \sup_{x,y \in M_j^*} \int_{M_j^*} \frac{G_P^M(x,z)|V(z)|G_P^M(z,y)}{G_P^M(x,y)} \,\mathrm{d}\mu(z) \right\} = 0. \tag{23}$$

(ii) V is a *semismall perturbation* of P in M if for some $x_0 \in M$ we have

$$\lim_{j\to\infty}\left\{\sup_{y\in M_j^*}\int_{M_j^*}\frac{G_P^M(x_0,z)|V(z)|G_P^M(z,y)}{G_P^M(x_0,y)}\,\mathrm{d}\mu(z)\right\}=0. \tag{24}$$

Recall that small perturbations are semismall [13]. For semismall perturbations we have

Theorem 2.6 ([13, 14, 15])**.** *Let P be a subcritical operator in M. Assume that $V = V_+ - V_-$ is a semismall perturbation of P^* in M satisfying $V_- \neq 0$, where $V_\pm(x) = \max\{0, \pm V(x)\}$.*

Then there exists $\alpha_0 > 0$ such that $P_\alpha := P + \alpha V$ is subcritical in M for all $0 \le \alpha < \alpha_0$ and critical for $\alpha = \alpha_0$.

Moreover, let φ be the ground state of $P + \alpha_0 V$ and let y_0 be a fixed reference point in M_1. Then for any $0 \le \alpha < \alpha_0$

$$\varphi \asymp G_{P_\alpha}^M(\cdot, y_0) \qquad \text{in } M_1^*.$$

3. The symmetric case

In this section we prove the following theorem.

Theorem 3.1. *Let the subcritical operator P_+ and the critical operator P_0 be symmetric in M. Assume that A_+ and A_0, the sections on M of* End(TM)*, and the weights m_+ and m_0, corresponding to P_+ and P_0, respectively, satisfy the following matrix inequality*

$$m_+(x)A_+(x) \le C m_0(x)A_0(x) \qquad \text{for a.e. } x \in M, \tag{25}$$

where C is a positive constant. Assume further that condition (16) holds true. Then

$$\lim_{t\to\infty}\frac{k_{P_+}^M(x,y,t)}{k_{P_0}^M(x,y,t)} = 0 \tag{26}$$

locally uniformly in $M \times M$.

Proof. By Theorem 2.3, we may assume that $\lambda_0(P_+, M) = 0$.

Assume to the contrary that for some $x_0, y_0 \in M$ there exists a sequence $\{t_n\}$ such that $t_n \to \infty$ and

$$\lim_{n\to\infty}\frac{k_{P_+}^M(x_0,y_0,t_n)}{k_{P_0}^M(x_0,y_0,t_n)} = K > 0. \tag{27}$$

Consider the sequence of functions $\{u_n\}_{n=1}^\infty$ defined by

$$u_n(x,s) := \frac{k_{P_+}^M(x,y_0,t_n+s)}{k_{P_0}^M(x_0,y_0,t_n)} \qquad x \in M,\ s \in \mathbb{R}.$$

We note that

$$u_n(x,s) = \frac{k_{P_+}^M(x,y_0,t_n+s)}{k_{P_+}^M(x_0,y_0,t_n)} \times \frac{k_{P_+}^M(x_0,y_0,t_n)}{k_{P_0}^M(x_0,y_0,t_n)}\,.$$

Therefore, by assumption (27) and Remark 2 it follows that we may subtract a subsequence which we rename by $\{u_n\}$ such that

$$\lim_{n\to\infty} u_n(x,s) = u_+(x,s),$$

where $u_+ \in \mathcal{H}_{P_+}(M \times \mathbb{R})$ and $u_+ \gneqq 0$.

On the other hand,

$$v_n(x) := \frac{k_{P_+}^M(x, y_0, t_n)}{k_{P_0}^M(x_0, y_0, t_n)} = u_n(x, s) \frac{k_{P_+}^M(x, y_0, t_n)}{k_{P_+}^M(x, y_0, t_n + s)}\,.$$

By our assumption, $\lambda_0(P_+, M) = 0$, therefore Lemma 2.2 implies that

$$\lim_{n\to\infty} \frac{k_{P_+}^M(x, y_0, t_n)}{k_{P_+}^M(x, y_0, t_n + s)} = 1.$$

Therefore,

$$\lim_{n\to\infty} v_n(x) = \lim_{n\to\infty} u_n(x, s) = u_+(x, s),$$

and u_+ does not depend on s, and hence u_+ is a positive solution of the elliptic equation $P_+u = 0$ in M and we have

$$\lim_{n\to\infty} \frac{k_{P_+}^M(x, y_0, t_n)}{k_{P_0}^M(x_0, y_0, t_n)} = u_+(x). \tag{28}$$

On the other hand, by Remark 1 we have

$$\lim_{n\to\infty} \frac{k_{P_0}^M(x, y_0, t_n)}{k_{P_0}^M(x_0, y_0, t_n)} = \frac{\varphi(x)}{\varphi(x_0)} =: u_0(x), \tag{29}$$

where φ is the ground state of P_0.

Combining (28) and (29), we obtain

$$\lim_{n\to\infty} \frac{k_{P_+}^M(x, y_0, t_n)}{k_{P_0}^M(x, y_0, t_n)} = \lim_{n\to\infty} \left\{ \frac{\frac{k_{P_+}^M(x, y_0, t_n)}{k_{P_0}^M(x_0, y_0, t_n)}}{\frac{k_{P_0}^M(x, y_0, t_n)}{k_{P_0}^M(x_0, y_0, t_n)}} \right\} = \frac{u_+(x)}{u_0(x)}. \tag{30}$$

On the other hand, by assumption (16) and the parabolic Harnack inequality there exists a positive constant C_1 which depends on P_+, P_0, y_0, y_1 such that

$$\begin{aligned} C_1^{-1} k_{P_+}^M(x, y_0, t-1) &\leq k_{P_+}^M(x, y_1, t) \\ &\leq C k_{P_0}^M(x, y_1, t) \leq C C_1 k_{P_0}^M(x, y_0, t+1) \quad \forall x \in M, t > T(x). \end{aligned} \tag{31}$$

Moreover, by Lemma 2.2 we have

$$\lim_{t\to\infty} \frac{k_{P_+}^M(x, y_0, t-1)}{k_{P_+}^M(x, y_0, t)} = 1, \quad \text{and} \quad \lim_{t\to\infty} \frac{k_{P_0}^M(x, y_0, t+1)}{k_{P_0}^M(x, y_0, t)} = 1 \qquad \forall x \in M. \tag{32}$$

Therefore, (31) and (32) imply that there exists $C_0 > 0$ such that

$$k_{P_+}^M(x, y_0, t) \leq C_0 k_{P_0}^M(x, y_0, t) \qquad \forall x \in M, t > T(x). \tag{33}$$

Consequently, (30) and (33) imply that

$$u_+(x) \leq C_0 u_0(x) = \tilde{C}_0 \varphi(x) \qquad \forall x \in M.$$

Therefore, using (25) we obtain

$$(u_+)^2(x) m_+(x) A_+(x) \leq C_2 \varphi^2(x) m_0(x) A_0(x) \qquad \text{for a.e. } x \in M, \tag{34}$$

where $C_2 > 0$ is a positive constant. Thus, Theorem 2.4 implies that P_+ is critical in M which is a contradiction. The last statement of the theorem follows from the parabolic Harnack inequality and parabolic regularity. □

By the generalized maximum principle, assumption (16) in Theorem 3.1 is satisfied with $C = 1$ if $P_+ = P_0 + V$, where P_0 is a critical operator on M and V is any *nonnegative* potential. Note that if the potential is in addition nontrivial, then P_+ is indeed subcritical in M. Therefore, we have

Corollary 1. *Let P_0 be a symmetric operator which is critical in M, and let $P_+ := P_0 + V$, where V is a nonzero nonnegative potential. Then*

$$\lim_{t\to\infty} \frac{k^M_{P_+}(x,y,t)}{k^M_{P_0}(x,y,t)} = 0 \tag{35}$$

locally uniformly in $M \times M$.

Remark 3. The pointwise limit (26) of Theorem 3.1 leads to a normwise limit of the type (8) in suitably chosen functional spaces. Let us assume that the initial data u_0 of (1) lie in the space $L^1_0(M)$ of compactly supported integrable functions on M equipped with the usual L^1-norm. Since e^{-P_+t} and e^{-P_0t} are positivity-preserving under the hypotheses of Theorem 3.1, we can restrict ourselves to $u_0 \geq 0$. For any $x \in M$, we have

$$\mathrm{e}^{-P_+t}u_0(x) = \int_M k_+(x,y,t)\, u_0(y)\, \mathrm{d}\mu(y) \leq \left\{ \sup_{y\in \mathrm{supp}(u_0)} \frac{k_+(x,y,t)}{k_0(x,y,t)} \right\} \mathrm{e}^{-P_0t}u_0(x)\,.$$

Consequently, for any compact set $K \Subset M$, we arrive at

$$\frac{\|\mathrm{e}^{-P_+t}\|_{L^1_0(M)\to L^\infty(K)}}{\|\mathrm{e}^{-P_0t}\|_{L^1_0(M)\to L^\infty(K)}} \leq \sup_{x\in K,\ y\in \mathrm{supp}(u_0)} \frac{k_+(x,y,t)}{k_0(x,y,t)} \xrightarrow[t\to 0]{} 0$$

by Theorem 3.1.

4. Davies' conjecture and Conjecture 1

In the present section we discuss the nonsymmetric case. We study two cases where Davies' conjecture imply Conjecture 1. First, we show that in the nonsymmetric case, the result of Corollary 1 for positive perturbations of a critical operator P_0 still holds provided that the validity of Davies' conjecture (Conjecture 2) is assumed instead of the symmetry hypothesis. More precisely, we have

Theorem 4.1. *Let P_0 be a critical operator in M, and let $P_+ = P_0 + V$, where V is any nonzero nonnegative potential on M. Assume that Davies' conjecture (Conjecture 2) holds true for both $k^M_{P_0}$ and $k^M_{P_+}$. Then*

$$\lim_{t\to\infty} \frac{k^M_{P_+}(x,y,t)}{k^M_{P_0}(x,y,t)} = 0 \tag{36}$$

locally uniformly in $M \times M$.

Proof. By Theorem 2.3, we may assume that $\lambda_0(P_+, M) = 0$.

Assume to the contrary that for some $x_0, y_0 \in M$ there exists a sequence $\{t_n\}$ such that $t_n \to \infty$ and

$$\lim_{n\to\infty} \frac{k^M_{P_+}(x_0,y_0,t_n)}{k^M_{P_0}(x_0,y_0,t_n)} = K > 0. \tag{37}$$

Consider the functions v_+ and v_0 defined by

$$v_+(x,t) := \frac{k^M_{P_+}(x,y_0,t)}{k^M_{P_+}(x_0,y_0,t)}, \quad v_0(x,t) := \frac{k^M_{P_0}(x,y_0,t)}{k^M_{P_0}(x_0,y_0,t)} \qquad x \in M, t > 0.$$

By our assumption,

$$\lim_{t\to\infty} v_+(x,t) = u_+(x), \qquad \lim_{t\to\infty} v_0(x,t) = u_0(x),$$

where $u_+ \in \mathcal{C}_{P_+}(M)$ and $u_0 \in \mathcal{C}_{P_0}(M)$.

On the other hand, by the generalized maximum principle

$$\frac{k^M_{P_+}(x,y_0,t)}{k^M_{P_0}(x,y_0,t)} \leq 1. \tag{38}$$

Therefore,

$$\frac{k^M_{P_+}(x_0,y_0,t_n)}{k^M_{P_0}(x_0,y_0,t_n)} \times \frac{\frac{k^M_{P_+}(x,y_0,t_n)}{k^M_{P_+}(x_0,y_0,t_n)}}{\frac{k^M_{P_0}(x,y_0,t_n)}{k^M_{P_0}(x_0,y_0,t_n)}} = \frac{k^M_{P_+}(x,y_0,t_n)}{k^M_{P_0}(x,y_0,t_n)} \leq 1. \tag{39}$$

Letting $n \to \infty$ we obtain

$$Ku_+(x) \leq u_0(x) \qquad x \in M. \tag{40}$$

It follows that $v(x) := u_0(x) - Ku_+(x)$ is a nonnegative supersolution of the equation $P_0 u = 0$ in M which is not a solution. In particular, $v \neq 0$. By the strong maximum principle $v(x)$ is a strictly positive supersolution of the equation $P_0 u = 0$ in M which is not a solution. This contradicts the criticality of P_0 in M. □

The second result concerns semismall perturbations.

Theorem 4.2. *Let P_+ and $P_0 = P_+ + V$ be a subcritical operator and a critical operator in M, respectively. Suppose that V is a semismall perturbation of the operator P_+^* in M. Assume further that Davies' conjecture (Conjecture 2) holds true for both $k^M_{P_0}$ and $k^M_{P_+}$ and that (16) holds true. Then*

$$\lim_{t\to\infty} \frac{k^M_{P_+}(x,y,t)}{k^M_{P_0}(x,y,t)} = 0 \tag{41}$$

locally uniformly in $M \times M$.

Proof. The first part of the proof is similar to the corresponding part in the proof of Theorem 4.1. For completeness we repeat it.

By Theorem 2.3, we may assume that $\lambda_0(P_+, M) = 0$.

Assume to the contrary that for some $x_0, y_0 \in M$ there exists a sequence $\{t_n\}$ such that $t_n \to \infty$ and

$$\lim_{n\to\infty} \frac{k^M_{P_+}(x_0,y_0,t_n)}{k^M_{P_0}(x_0,y_0,t_n)} = K > 0. \tag{42}$$

Consider the functions v_+ and v_0 defined by

$$v_+(x,t) := \frac{k^M_{P_+}(x,y_0,t)}{k^M_{P_+}(x_0,y_0,t)}, \quad v_0(x,t) := \frac{k^M_{P_0}(x,y_0,t)}{k^M_{P_0}(x_0,y_0,t)} \qquad x \in M, t > 0. \tag{43}$$

By our assumption,

$$\lim_{t\to\infty} v_+(x,t) = u_+(x), \qquad \lim_{t\to\infty} v_0(x,t) = u_0(x),$$

where $u_+ \in \mathcal{C}_{P_+}(M)$ and $u_0 \in \mathcal{C}_{P_0}(M)$.

On the other hand, by assumption (16) we have for $t > T(x)$

$$\frac{k_{P_+}^M(x,y_0,t)}{k_{P_0}^M(x,y_0,t)} = \frac{k_{P_+}^M(x,y_1,t)}{k_{P_0}^M(x,y_1,t)} \times \frac{\frac{k_{P_+}^M(x,y_0,t)}{k_{P_+}^M(x,y_1,t)}}{\frac{k_{P_0}^M(x,y_0,t)}{k_{P_0}^M(x,y_1,t)}} \leq C\frac{k_{P_+}^M(x,y_0,t)}{k_{P_+}^M(x,y_1,t)} \times \frac{k_{P_0}^M(x,y_1,t)}{k_{P_0}^M(x,y_0,t)}\,. \tag{44}$$

By our assumption on Davies' conjecture, we have for a fixed x

$$\lim_{t\to\infty} \frac{k_{P_+}^M(x,y_0,t)}{k_{P_+}^M(x,y_1,t)} = \frac{u_+^*(y_0)}{u_+^*(y_1)}\,, \qquad \lim_{t\to\infty} \frac{k_{P_0}^M(x,y_1,t)}{k_{P_0}^M(x,y_0,t)} = \frac{u_0^*(y_1)}{u_0^*(y_0)}, \tag{45}$$

where u_+^* and u_0^* are positive solutions of the equation $P_+^*u = 0$ and $P_0^*u = 0$ in M, respectively. By the elliptic Harnack inequality there exists a positive constant C_1 (depending on P_+^*, P_0^*, y_0, y_1 but not on x) such that

$$\frac{u_+^*(y_0)}{u_+^*(y_1)} \leq C_1, \qquad \frac{u_0^*(y_1)}{u_0^*(y_0)} \leq C_1. \tag{46}$$

Therefore, (44) and (46) imply that

$$\frac{k_{P_+}^M(x,y_0,t_n)}{k_{P_0}^M(x,y_0,t_n)} \leq 2CC_1^2 \tag{47}$$

for n sufficiently large (which might depend on x).

Therefore,

$$\frac{k_{P_+}^M(x_0,y_0,t_n)}{k_{P_0}^M(x_0,y_0,t_n)} \times \frac{\frac{k_{P_+}^M(x,y_0,t_n)}{k_{P_+}^M(x_0,y_0,t_n)}}{\frac{k_{P_0}^M(x,y_0,t_n)}{k_{P_0}^M(x_0,y_0,t_n)}} = \frac{k_{P_+}^M(x,y_0,t_n)}{k_{P_0}^M(x,y_0,t_n)} \leq 2CC_1^2\,. \tag{48}$$

Letting $n \to \infty$ and using (42) and (43), we obtain

$$Ku_+(x) \leq 2CC_1^2 u_0(x) \qquad x \in M\,. \tag{49}$$

On the other hand, since V is a semismall perturbation of P_+^* in M, Theorem 2.6 implies that $u_0(x) \asymp G_{P_+}^M(x,y_0)$ in $M\setminus\overline{B(y_0,\delta)}$, with some positive δ. Consequently,

$$u_+(x) \leq C_2 G_{P_+}^M(x,y_0) \qquad x \in M \setminus \overline{B(y_0,\delta)} \tag{50}$$

for some $C_2 > 0$. In other words, u_+ is a global positive solution which has minimal growth in a neighborhood of infinity in M. Therefore u_+ is a ground state of the equation $P_+u = 0$ in M, but this contradicts the subcriticality of P_+ in M. $\square$

5. On the equivalence of heat kernels

In this section we study a general question concerning the equivalence of heat kernels which in turn will give sufficient conditions for the validity of the boundedness assumption (16) which is assumed in theorems 2.3, 3.1 and 4.2.

Definition 5.1. Let P_i, $i = 1, 2$, be two elliptic operators on M satisfying $\lambda_0(P_i, M) \geq 0$ for $i = 1, 2$. We say that the heat kernels $k_{P_1}^M(x, y, t)$ and $k_{P_2}^M(x, y, t)$ are *equivalent* (respectively, *semiequivalent*) if $k_{P_1}^M \asymp k_{P_2}^M$ on $M \times M \times (0, \infty)$ (respectively, $k_{P_1}^M(\cdot, y_0, \cdot) \asymp k_{P_2}^M(\cdot, y_0, \cdot)$ on $M \times (0, \infty)$ for some fixed $y_0 \in M$).

There is an intensive literature dealing with (almost optimal) conditions under which two positive (minimal) Green functions are equivalent or semiequivalent (see [**2**, **13**, **14**, **17**] and the references therein). On the other hand, sufficient conditions for the equivalence of heat kernels are known only in a few cases (see [**10**, **11**, **23**]). In particular, it seems that the answer to the following conjecture is not known.

Conjecture 3. *Let P_1 and P_2 be two subcritical operators of the form (10) which are defined on a Riemannian manifold M such that $P_1 = P_2$ outside a compact set in M. Then $k_{P_1}^M$ and $k_{P_2}^M$ are equivalent.*

It is well known that certain 3-G inequalities imply the equivalence of Green functions, and the notions of small and semismall perturbations is based on this fact. Moreover, small (respectively, semismall) perturbations are sufficient conditions and in some sense also necessary conditions for the equivalence (respectively, semiequivalence) of the Green functions [**13**, **14**, **17**]. We introduce an analog definition for heat kernels (cf. [**23**]).

Definition 5.2. Let P be a subcritical operator in M. We say that a potential V is a *k-bounded perturbation* (respectively, *k-semibounded perturbation*) with respect to the heat kernel $k_P^M(x, y, t)$ if there exists a positive constant C such that the following 3-k inequality is satisfied:

$$\int_0^t \int_M k_P^M(x, z, t - s)|V(z)|k_P^M(z, y, s)\, \mathrm{d}\mu(z)\, \mathrm{d}s \leq C k_P^M(x, y, t) \tag{51}$$

for all $x, y \in M$ (respectively, for a fixed $y \in M$, and all $x \in M$) and $t > 0$.

The following result shows that the notion of k-(semi)boundedness is naturally related to the (semi)equivalence of heat kernels.

Theorem 5.3. *Let P be a subcritical operator in M, and assume that the potential V is k-bounded perturbation (respectively, k-semibounded perturbation) with respect to the heat kernel $k_P^M(x, y, t)$. Then there exists $c > 0$ such that for any $|\varepsilon| < c$ the heat kernels $k_{P+\varepsilon V}^M(x, y, t)$ and $k_P^M(x, y, t)$ are equivalent (respectively, semiequivalent).*

Proof. Consider the iterated kernels

$$k_P^{(j)}(x, y, t) := \begin{cases} k_P^M(x, y, t) & j = 0, \\ \int_0^t\int_M k_P^{(j-1)}(x, z, t - s)V(z)k_P^M(z, y, s)\, \mathrm{d}\mu(z)\, \mathrm{d}s & j \geq 1. \end{cases}$$

Using (51) and an induction argument, it follows that

$$\sum_{j=0}^{\infty} |\varepsilon|^j |k_P^{(j+1)}(x, y, t)|$$
$$\leq \left(1 + C|\varepsilon| + C^2|\varepsilon|^2 + \dots\right) k_P^M(x, y, t) = \frac{1}{1 - C|\varepsilon|}\, k_P^M(x, y, t)$$

provided that $|\varepsilon| < C^{-1}$. Consequently, for such ε the Neumann series

$$\sum_{j=0}^{\infty}(-\varepsilon)^j k_P^{(j+1)}(x,y,t)$$

converges locally uniformly in $M\times M\times\mathbb{R}_+$ to $k^M_{P+\varepsilon V}(x,y,t)$ which in turn implies that Duhamel's formula

$$k^M_{P+\varepsilon V}(x,y,t) = k^M_P(x,y,t) - \varepsilon\int_0^t\int_M k^M_P(x,z,t-s)V(z)k^M_{P+\varepsilon V}(z,y,s)\,\mathrm{d}\mu(z)\,\mathrm{d}s \tag{52}$$

is valid. Moreover, we have

$$k^M_{P+\varepsilon V}(x,y,t) \le \frac{1}{1-C|\varepsilon|}\,k^M_P(x,y,t).$$

The lower bound

$$C_1 k^M_P(x,y,t) \le k^M_{P+\varepsilon V}(x,y,t)$$

(for $|\varepsilon|$ small enough) follows from the upper bound using (52) and (51). □

Corollary 2. *Assume that P and V satisfy the conditions of Theorem 5.3, and suppose further that V is nonnegative. Then there exists $c>0$ such that for any $\varepsilon > -c$ the heat kernels $k^M_{P+\varepsilon V}(x,y,t)$ and $k^M_P(x,y,t)$ are equivalent (respectively, semiequivalent).*

Proof. By Theorem 5.3 the heat kernels $k^M_{P+\varepsilon V}(x,y,t)$ and $k^M_P(x,y,t)$ are equivalent (respectively, semiequivalent) for any $|\varepsilon|<c$. Recall that by the generalized maximum principle,

$$k^M_{P+\varepsilon V}(x,y,t) \le k^M_P(x,y,t) \qquad \forall\varepsilon>0.$$

On the other hand, using also Lemma 2.1, we obtain the desired equivalence also for $\varepsilon \ge c$. □

Theorem 5.4. *Let P_0 be a critical operator in M. Assume that $V = V_+ - V_-$ is a potential such that $V_\pm \ge 0$ and $P_+ := P_0 + V$ is subcritical in M.*

Assume further that V_- is k-semibounded perturbation with respect to the heat kernel $k^M_{P_+}(x,y_1,t)$. Then condition (16) is satisfied uniformly in x. That is, there exist positive constants C and T such that

$$k^M_{P_+}(x,y_1,t) \le C k^M_{P_0}(x,y_1,t) \qquad \forall x\in M, t>T. \tag{53}$$

Proof. By Corollary 2, the heat kernels $k^M_{P_+}(x,y_1,t)$ and $k^M_{P_++V_-}(x,y_1,t)$ are semiequivalent. Note that $P_+ + V_- = P_0 + V_+$, Therefore we have

$$C^{-1}k^M_{P_+}(x,y_1,t) \le k^M_{P_0+V_+}(x,y_1,t) \le k^M_{P_0}(x,y_1,t) \qquad \forall x\in M, t>0. \tag{54}$$

□

Acknowledgments

Part of this research was done while D. K. was visiting the Technion. D. K. would like to thank the Technion for the kind hospitality. The work of D. K. was partially supported by the Czech Ministry of Education, Youth and Sports within the project LC06002. M. F. and Y. P. acknowledge the support of the Israel Science Foundation (grant No. 587/07) founded by the Israeli Academy of Sciences and Humanities. M. F. acknowledges also the support of the Israel Science Foundation

(grant No. 419/07) founded by the Israeli Academy of Sciences and Humanities. M. F. was also partially supported by a fellowship of the UNESCO fund.

Remark 4 (Added on 17.5.2010)**.** After the paper has been published, we realized that the generalized principal eigenvalue is characterized by the following formula

$$\lim_{t\to\infty}\frac{\log k_P^M(x,y,t)}{t} = -\lambda_0(P,M). \tag{55}$$

The above characterization of λ_0 is well-known in the symmetric case, see for example [**7**, Theorem 10.24]. The needed upper bound for the validity of (55) for general elliptic operators of the form (10) follows directly from Theorem 1.1, while the lower bound follows from the large time behavior of the heat kernel in a smooth bounded domain using a standard exhaustion argument (cf. the proof of [**7**, Theorem 10.24]).

Consequently, if P_0 is a critical operator in M, and P_+ is a subcritical operator in M satisfying $\lambda_+ := \lambda_0(P_+, M) > 0$, then Conjecture 1 holds true without any further assumption (cf. Theorem 2.3).

References

[1] S. Agmon, *On positivity and decay of solutions of second order elliptic equations on Riemannian manifolds*, in "Methods of Functional Analysis and Theory of Elliptic Equations" (ed. D. Greco), Liguori, Naples, (1983), 19–52. MR 0819005

[2] A. Ancona, *First eigenvalues and comparison of Green's functions for elliptic operators on manifolds or domains*, J. Anal. Math., **72** (1997), 45–92. MR 1482989

[3] J-P Anker, P. Bougerol and T. Jeulin, *The infinite Brownian loop on symmetric space*, Rev. Mat. Iberoamericana, **18** (2002), 41–97. MR 1924687

[4] M. Barlow, T. Coulhon and A. Grigor'yan, *Manifolds and graphs with slow heat kernel decay*, Invent. Math., **144** (2001), 609–649. MR 1833895

[5] P. Collet, S. Martínez and J. San Martín, *Ratio limit theorems for a Brownian motion killed at the boundary of a Benedicks domain*, Ann. Probab., **27** (1999), 1160–1182. MR 1733144

[6] E. B. Davies, *Non-Gaussian aspects of heat kernel behaviour*, J. London Math. Soc. (2), **55** (1997), 105–125. MR 1423289

[7] A. Grigor'yan, "Heat Kernel and Analysis on Manifolds", AMS/IP Studies in Advanced Mathematics, 47. American Mathematical Society, Providence, RI; International Press, Boston, MA, 2009. MR MR2569498

[8] G. Kozma, private communication, and a talk in a probability seminar at UBC, 2006.

[9] D. Krejčiřík and E. Zuazua, *The Hardy inequality and the heat equation in twisted tubes*, to appear in J. Math. Pures Appl. (http://dx.doi.org/10.1016/j.matpur.2010.02.006).

[10] V. Liskevich and Y. Semenov, *Two-sided estimates of the heat kernel of the Schrödinger operator*, Bull. London Math. Soc., **30** (1998), 596–602. MR 1642818

[11] P. D. Milman and Yu. A. Semenov, *Heat kernel bounds and desingularizing weights*, J. Funct. Anal., **202** (2003), 1–24. MR 1994762

[12] M. Murata, *Positive solutions and large time behaviors of Schrödinger semigroups, Simon's problem*, J. Funct. Anal., **56** (1984), 300–310. MR 0743843

[13] M. Murata, *Semismall perturbations in the Martin theory for elliptic equations*, Israel J. Math., **102** (1997), 29–60. MR 1489100

[14] Y. Pinchover, *Criticality and ground states for second-order elliptic equations*, J. Differential Equations, **80** (1989), 237–250. MR 1011149

[15] Y. Pinchover, *On criticality and ground states of second order elliptic equations, II*, J. Differential Equations, **87** (1990), 353–364. MR 1072906

[16] Y. Pinchover, *Large time behavior of the heat kernel and the behavior of the Green function near criticality for nonsymmetric elliptic operators*, J. Funct. Anal., **104** (1992), 54–70. MR 1152459

[17] Y. Pinchover, *Maximum and anti-maximum principles and eigenfunctions estimates via perturbation theory of positive solutions of elliptic equations*, Math. Ann., **314** (1999), 555–590. MR 1704549

[18] Y. Pinchover, *Large time behavior of the heat kernel*, J. Funct. Anal., **206** (2004), 191–209. MR 2024351

[19] Y. Pinchover, *On Davies' conjecture and strong ratio limit properties for the heat kernel*, in "Potential Theory in Matsue", Proceedings of the International Workshop on Potential Theory, 2004 (eds. H. Aikawa, et al.), Advanced Studies in Pure Mathematics **44**, Mathematical Society of Japan, Tokyo, (2006), 339–352. MR 2279767

[20] Y. Pinchover, *A Liouville-type theorem for Schrödinger operators*, Comm. Math. Phys., **272** (2007), 75–84. MR 2291802

[21] R. G. Pinsky, "Positive Harmonic Function and Diffusion", Cambridge University Press, Cambridge, 1995. MR 1326606

[22] B. Simon, *Schrödinger semigroups*, Bull. Amer. Math. Soc. (N.S.), **7** (1982), 447–526. MR 0670130

[23] Q. S. Zhang, *A sharp comparison result concerning Schrödinger heat kernels*, Bull. London Math. Soc., **35** (2003), 461–472. MR 1978999

E-mail address: `martin.fraas@gmail.com`

E-mail address: `krejcirik@ujf.cas.cz`

E-mail address: `pincho@techunix.technion.ac.il`